

TALK IN POSTDOC SEMINAR FALL '15 (?)

Liouville action:

Σ - Riemann surface

$\varphi(z, \bar{z})$ real field

φ is independent

Liouville action: $S(\varphi) = \int_{\Sigma} \left(|\varphi_z|^2 + e^{\varphi} \right) \frac{dx dy}{2}$

(not glob. well-defined)

Euler-Lagrange: $2\varphi_{z\bar{z}} - e^{\varphi} = 0$

classical

Liouville equation:

$$\varphi_{z\bar{z}} = \frac{1}{2} e^{\varphi}$$

$\Rightarrow$ metric $g = e^{\varphi} |dz|^2$

has constant curvature = -1 .
Standard solutions: $\cdot H \rightarrow \varphi = -2 \log \frac{1}{|z|} \Rightarrow g = \frac{4}{|z|^2} |dz|^2$

upper half plane

Transformation law: $z \mapsto f(z) \Rightarrow g = \frac{4}{|f'(z)|^2} |dz|^2$

conformal disk

(complete metrics)

$\Rightarrow g \mapsto e^{\varphi(f(z))} |f'(z)|^2$

$= e^{\varphi(f(z))} |f'|^2 dz d\bar{z}$

$\Rightarrow \varphi_z \mapsto \varphi(f(z)) + \log |f'|^2$

Stress-energy tensor?

$(\varphi_{z\bar{z}})_z = \frac{1}{2} \varphi_z e^{\varphi} = \frac{1}{2} \varphi_z \cdot 2\varphi_{z\bar{z}}$

$(\varphi_{z\bar{z}})_{\bar{z}} = \left(\frac{1}{2} |\varphi_z|^2 \right)_{\bar{z}}$

$\frac{\partial}{\partial \bar{z}} \left(\varphi_{zz} - \frac{1}{2} |\varphi_z|^2 \right) = 0$

$(2,0)$ ^{Te} cpt of stress-energy

trace = (1,1) cpt should vanish?

Transformation of T_ϕ :

$$\begin{aligned} & (\phi(f(z)) + \log|f'|^2)_{zz} - \frac{1}{2} \left[(\phi(f(z)) + \log|f'|^2)' \right]^2 \\ &= f'' \phi_z + (f')^2 \phi_{zz} + (\log f')'' \\ & \quad - \frac{1}{2} \left[f' \phi_z + (\log f')' \right]^2 \end{aligned}$$

$$\phi(f)_z = f' \phi_z$$

$$\begin{aligned} \phi(f)_{zz} &= f'' \phi_z \\ & \quad + (f')^2 \phi_{zz} \end{aligned}$$

$$= \cancel{f'' \phi_z} + (f')^2 \left[\phi_{zz} - \frac{1}{2} \phi_z^2 \right] + (\log f')''$$

$$= \cancel{f' \phi_z \frac{f''}{f'}} - \frac{1}{2} \left[(\log f')' \right]^2$$

$$= (f')^2 T_\phi(f(z)) + (\log f')'' - \frac{1}{2} \left[(\log f')' \right]^2$$

Schwarzian: $S(f) = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$

if $f \in \text{PGL}(2, \mathbb{C})$

$$z \mapsto \frac{az+b}{cz+d}$$

$$(c \neq 0)$$

then $S(f) = 0$.

" T_ϕ transforms as a projective connection."

Physics: Try to quantize Liouville theory.
Stay on $\mathbb{R}$ so have global coord. z .
Conformal Ward identities.

Math: Study Liouville equation
on $\mathbb{C} \setminus \{z_1, \dots, z_n\}$ with prescribed singularities

$$z_1, \dots, z_n, \dots$$

$0 < \alpha_i < 1 \Rightarrow$ cone points of angle $2\pi(1-\alpha_i)$.

$$e^\varphi \sim \frac{r^{2(\alpha_i-1)}}{(1-r^{2\alpha_i})^2}$$

= pull back via z^{α_i} of $\mathbb{D}$. (improper)

$\alpha_i = 1 \Rightarrow e^\varphi \sim \frac{1}{|\log r|^2 r^2}$ Hyperbolic
conf.

= pull back via $\log z$ of $\mathbb{H}$.
(complete)

Fact: Assume $\sum \alpha_i > 2$ (Goursat-Benoist).

Then Liouville eq'n is solvable w/ above
asymptotics!

Case $\alpha_i = 1$, this is the Uniformization
Theorem.

Statement:

Parts:

1st: Fuchsian equations

2nd: Variational methods

3rd: Ricci flow.

Kayden Wanner, Targem.

Example: $\Sigma = \mathbb{P}^2 - \{0, 1, \infty\}$

How to uniformize $J: \mathbb{H} \rightarrow \Sigma$?
 $\uparrow$
 multi-valued.

$$z(1-z) \frac{d^2 u}{dz^2} + \left(\frac{1}{4}\right) u = 0.$$

$$J^{-1} \cdot u = {}_2F_1\left(0, \frac{1}{2}, 1, z\right) \quad \text{or} \quad {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, 1-z\right).$$

$$= \Sigma$$

$$\tau = J^{-1} = i \cdot \frac{{}_2F_1(1, 1, 1, 1-z)}{{}_2F_1(1, 1, 1, z)}$$

$$J = \lambda(\tau) = \left(\frac{\theta_2(0, \tau)}{\theta_3(0, \tau)} \right)^4$$

$$q = e^{\pi i \tau} \quad \theta_2^{10, \tau} = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}$$

$$\theta_3^{10, \tau} = \sum_{n=-\infty}^{\infty} q^{n^2}$$